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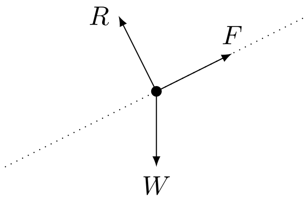
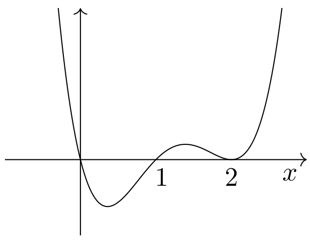
1501. The normal reaction acts perpendicular to the roof, the friction parallel to it.



(a) Perpendicular to the roof, $R = \frac{\sqrt{3}}{2}W$.
(b) Parallel to the roof, $F = \frac{1}{2}W$.
(c) This can be written down without calculation: the total contact force must be W.

1502. The quartic $y = x(x-1)(x-2)^2$ is the minimal example of such a polynomial. It crosses the x axis at $x = 0$ and $x = 1$, and is tangent to it at $x = 2$.



1503. According to the quadratic formula, if p and q are to be integers, the second root must be $\frac{1}{2}(1-\sqrt{5})$. By the factor theorem, the quadratic is

$$\left(x - \tfrac{1}{2}(1+\sqrt{5})\right)\left(x - \tfrac{1}{2}(1-\sqrt{5})\right) = 0.$$

Multiplying out gives $x^2 - x - 1 = 0$.

Alternative Method

The equation is monic, with $a = 1$. So, we need

$$\frac{-b+\sqrt{b^2-4c}}{2} = \frac{1+\sqrt{5}}{2}.$$

Equating rational parts, $b = -1$. Then $b^2 - 4c = 5$ gives $c = -1$. So, the equation is $x^2 - x - 1 = 0$.

1504. (a) Not binomial: the outcomes are all even.
(b) Since X_1, X_2 are independent, this is binomial: eight identical trials as opposed to just four. The distribution is $X_1 + X_2 \sim B(8, 0.5)$.
(c) Not binomial: there are negative outcomes.

1505. (a) We write the numerator as a multiple of the denominator plus a constant remainder:

$$\begin{aligned}&\frac{x^2+4}{x+1}\\ \equiv\ &\frac{(x-1)(x+1)+5}{x+1}\\ \equiv\ &\frac{(x-1)(x+1)}{x+1} + \frac{5}{x+1}\\ \equiv\ &x - 1 + \frac{5}{x+1}.\end{aligned}$$

(b) As $x \to \pm\infty$, the fraction becomes negligible. Hence, the curve approaches $y = x - 1$, which is therefore an oblique asymptote.

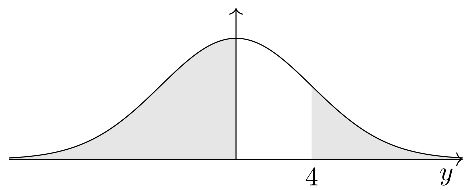
1506. The boundary equation is $u_n = 0$. Solving this,

$$\begin{aligned}4n^2 - 18n - 40 &= 0\\ \implies n &= \frac{18 \pm \sqrt{964}}{8}\\ &= -1.63..., 6.13...\end{aligned}$$

Hence, since the ordinal formula is a positive quadratic, the first positive term will be at $n = 7$. Its value is $u_7 = 30$.

1507. Statement (a) is false; $(1, 1, 0)$ is a counterexample.

1508. Solving algebraically, $Y^2 > 4Y$ is satisfied when $Y \in (-\infty, 0) \cup (4, \infty)$. So, the required probability is the area shaded:



Using symmetry and then the statistical facility on a calculator,

$$p = 0.5 + 0.3415... = 0.842 \text{ (3sf)}.$$

1509. We can split the exponential up using the relevant index law, and then divide:

$$\begin{aligned}e^{2x+3y}\frac{dy}{dx} &= 2\\ \implies e^{2x}e^{3y}\frac{dy}{dx} &= 2\\ \implies e^{3y}\frac{dy}{dx} &= 2e^{-2x}.\end{aligned}$$

1510. Vertical and horizontal *suvat*:

s_y	y	s_x	x
u_y	0	u_x	u
v_y		v_x	
a_y	$-g$	a_x	0
t	t	t	t

These give $y = -\frac{1}{2}gt^2$ and $x = ut$. Rearranging to $t = \frac{x}{u}$, we substitute for t. This gives

$$y = -\frac{gx^2}{2u^2}.$$

Adding $+c$ to translate the curve by vector $\binom{0}{c}$ gives the required result:

$$y = c - \frac{gx^2}{2u^2}.$$

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1511. This is an arithmetic series with first term $a = 1$, common difference $d = 2$ and n terms. So, we have

$$\begin{aligned} &\tfrac{1}{2}n\big[2 + 2(n-1)\big] = 100 \\ \implies &n^2 = 100. \end{aligned}$$

Therefore, $n = 10$.

1512. The forwards implication $\implies$ does not hold. For a counterexample, consider

$$x = \frac{-1-\sqrt{5}}{2}.$$

In the other direction, if we square both sides of the second statement,

$$\begin{aligned} &\sqrt{x} = x + 1 \\ \implies &x = x^2 + 2x + 1 \\ \implies &0 = x^2 + x + 1. \end{aligned}$$

Hence, the direction of implication is $\impliedby$.

1513. Since the roles of x and y have been reversed, the transformation is a reflection in the line $y = x$.

1514. This is a GP with common ratio $r = 4$. The first term, therefore, is $a = u_1 = \frac{1}{16}u_3 = \frac{3}{2}$. Quoting the ordinal formula for a GP:

$$\begin{aligned} u_n &= \tfrac{3}{2} \times 4^{n-1} \\ &\equiv \tfrac{3}{2} \times \big(2^2\big)^{n-1} \\ &\equiv \tfrac{3}{2} \times 2^{2n-2} \\ &\equiv 3 \times 2^{2n-3}, \text{ as required.} \end{aligned}$$

1515. The three points are collinear, lying on $y = 2x - 8$. Assume, for a contradiction, that they all lie on a parabola $y = ax^2 + bx + c$. Substituting for y to find intersections with the line $y = 2x - 8$, we get a quadratic equation in x with three distinct roots $x = 4, 5, 7$. This is impossible. Hence, no quadratic graph passes through these three points.

1516. (a) $360°$,
(b) $\frac{1}{3} \times 360° = 120°$,
(c) $\text{lcm}(180°, 120°) = 360°$,
(d) $\text{lcm}(180°, 90°) = 180°$.

1517. At an x intercept, $\text{f}(a) = 0$. The point $(a, 0)$ also satisfies $y^2 = \text{f}(x)$. So, the statement is true.

1518. (a) This is binomial: a set of $n_1 + n_2$ independent trials with constant probability of success p. Hence $X_1 + X_2 \sim B(n_1 + n_2, p)$.
(b) This is not binomial. For starters, its outcomes do not form a list of integers $\{0, 1, ..., n\}$.
(c) Using part (a), this is binomial. Since $X_1 + X_2$ is the number of successes in $n_1 + n_2$ trials, this variable $n_1 + n_2 - X_1 - X_2$ is the number of failures. Its distribution is $B(n_1 + n_2, 1 - p)$.

1519. Setting $\frac{dy}{dx} = 2x + b = 0$, the vertex of the parabola is at $x = -\frac{b}{2}$. We are told that $x > 0$, hence, $-\frac{b}{2} > 0$. Therefore $b < 0$. □

1520. Multiplying out,

$$\begin{aligned} &\int_0^4 x(x+1)(x+2)\,dx \\ = &\int_0^4 x^3 + 3x^2 + 2x\,dx \end{aligned}$$

Integrating, this is

$$\begin{aligned} &= \Big[\tfrac{1}{4}x^4 + x^3 + x^2\Big]_0^4 \\ &= (64 + 64 + 16) - (0) \\ &= 144, \text{ as required.} \end{aligned}$$

1521. Using the chain rule to differentiate,

$$\begin{aligned} &y = \big(1 + x^2\big)^{-1} \\ \implies &\frac{dy}{dx} = -2x\big(1 + x^2\big)^{-2}. \end{aligned}$$

Evaluating at $x = 1$ gives $m_{\text{tangent}} = -1/2$. The coordinate is $(1, 1/2)$, so the tangent has equation

$$y - \tfrac{1}{2} = -\tfrac{1}{2}(x - 1).$$

At Q, $y = 0$, which gives $Q : (2, 0)$.

— Alternative Method —

Using the diagram, the tangent also passes through the y intercept $y = 1$. So, as x progresses $\{0, 1, 2\}$, y progresses $\{1, 1/2, 0\}$. Hence, Q is at $(2, 0)$.

1522. (a) The second derivative is zero everywhere, so g is linear. Its gradient is 4. Substituting $(-1, 2)$ gives $\text{g}(x) = 4x + 6$.
(b) The equation is

$$\begin{aligned} &\text{g}(x) = \text{g}(1 - x) \\ \implies &4x + 6 = 4(1 - x) + 6 \\ \implies &x = \tfrac{1}{2}. \end{aligned}$$

— Alternative Method —

The line $y = \text{g}(x)$ and the line $y = \text{g}(1 - x)$ are reflections in $x = \frac{1}{2}$. So, they must intersect at $x = \frac{1}{2}$, and cannot intersect anywhere else.

1523. (a) This may or may not be true. The IQR doesn't take account of the magnitudes of the highest and lowest values, so it is possible that it will be unaffected by such a removal.

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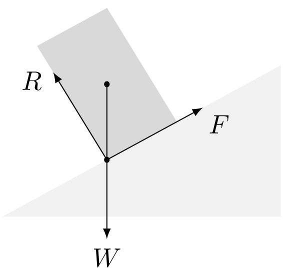
(b) This definitely holds. The standard deviation takes account of all data items. Hence, removal of four extreme values must reduce its value as a measure of spread.

1524. For SPs, we set the first derivative to zero:

$$\begin{aligned} 16x^3 - 16x &= 0 \\ \implies x &= -1, 0, 1. \end{aligned}$$

We test these values with the second derivative $48x^2 - 16$; the respective values are $32, -16, 32$. Hence, the SPs at $x = \pm 1$ are local minima. Their y coordinate is $y = -4$.

1525. The cross-section is a $2r$ (diameter in contact with the slope) by l rectangle. Since it is uniform, the centre of mass is in the middle. At the greatest angle of inclination, the cylinder is on the point of toppling, which means the reaction force and the weight must both have lines of action passing through the lowest point on the cylinder:



The acute angle between the reaction and the weight is given by $\tan\theta = \frac{2r}{l}$. This is also the angle of inclination, so $\theta = \arctan\frac{2r}{l}$, as required.

1526. Substituting $(k, 0)$, we require

$$\begin{aligned} 0 &= \frac{k^2}{1+k} + k. \\ \implies 0 &= k^2 + k(1+k) \\ \implies 0 &= k(2k+1) \\ \implies k &= 0, -\tfrac{1}{2}. \end{aligned}$$

1527. Since $x^{\frac{5}{3}} = \left(x^{\frac{5}{6}}\right)^2$, this is a quadratic in $x^{\frac{5}{6}}$. The formula gives

$$\begin{aligned} & x^{\frac{5}{3}} - 275x^{\frac{5}{6}} + 7776 = 0 \\ \implies & x^{\frac{5}{6}} = \frac{275 \pm \sqrt{275^2 - 4 \cdot 7776}}{2} \\ \implies & x^{\frac{5}{6}} = 32 \quad\text{or}\quad x^{\frac{5}{6}} = 243 \\ \implies & x = 32^{\frac{6}{5}} \quad\text{or}\quad x = 243^{\frac{6}{5}} \\ \implies & x = 64 \quad\text{or}\quad x = 729. \end{aligned}$$

1528. (a) The accelerations are given by $mg - \frac{1}{4}mg = ma$ and $2mg - \frac{1}{4}mg = 2ma$. These yield $a_1 = \frac{3}{4}g$ and $a_2 = \frac{7}{8}g$. Since the initial conditions are the same for the two balls, we can use relative acceleration, which is $a = \frac{7}{8}g - \frac{3}{4}g = \frac{1}{8}g$. With this value, the distance between them is

$$d = \tfrac{1}{2} \cdot \tfrac{1}{8}gt^2 = \frac{gt^2}{16}.$$

(b) Setting $d = 100$, we get $t = 12.8$ s (3sf).

(c) At this time, the heavier ball is modelled to be travelling at $v = 12.8 \times \frac{7}{8}g > 100$ m/s. Assuming air resistance to remain constant at such speeds is certainly inaccurate.

1529. The value of the mean gives

$$\frac{0 \times 12 + 1 \times 14 + 2a + 3b}{12 + 14 + a + b} = 2,$$

which simplifies to $b = 38$. Then, we can use the formula for the variance, with $n = 64 + a$:

$$\begin{aligned} s^2 &= \frac{\sum x^2 - n\bar{x}^2}{n} \\ \therefore \quad 1 &= \frac{14 + 4a + 9 \times 38 - (64+a)2^2}{64 + a} \\ \implies 64 + a &= 356 + 4a - 4(64 + a) \\ \implies a &= 36. \end{aligned}$$

1530. Setting up generic curves $y = ax^2 + bx + c$ and $y = px^3 + qx^2 + rx + s$, where $a, p \neq 0$, we can solve for intersections with

$$px^3 + (q-a)x^2 + (r-b)x + s - r = 0.$$

Since $p \neq 0$, this is a cubic equation. And, being a polynomial of odd degree, it must have at least one root. Hence, the curves must intersect. □

1531. (a) If these equations are to have no simultaneous solutions, then they must be a pair of distinct parallel lines. Equating the gradients, we get $\frac{b}{2} = 1$, so $b = 1$, as required.

(b) All values of a are permitted, except $a = 5$ in which case the lines are identical and there are infinitely many solutions. So, $a \in \mathbb{R} \setminus \{5\}$.

1532. The position vector of a midpoint is the mean of the two position vectors. So, the position vector of M is $\mathbf{m} = \frac{1}{2}(\mathbf{a} + \mathbf{b})$ and the position vector of N is $\mathbf{n} = \frac{1}{2}(\mathbf{c} + \mathbf{d})$. Hence, the midpoint P of MN has position vector

$$\begin{aligned} \mathbf{p} &= \tfrac{1}{2}(\mathbf{m} + \mathbf{n}) \\ &= \tfrac{1}{2}\left(\tfrac{1}{2}(\mathbf{a} + \mathbf{b}) + \tfrac{1}{2}(\mathbf{c} + \mathbf{d})\right) \\ &= \tfrac{1}{4}\left(\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d}\right). \end{aligned}$$

This is the mean, as required.

1533. The graph shown crosses the x axis. However, the algebraic fraction $\frac{1}{1+x^2}$ could never be zero, since its numerator is non-zero. Hence, it can't produce such a graph.

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1534. This is true. The equation is that of a circle centre $(-1,-1)$ with radius 1. Hence, it is tangent to the y axis at $(-1,0)$.

1535. Listing by number of blue edges:

$$\begin{aligned} 0 &: \quad \text{RRRRR} \\ 1 &: \quad \text{BRRRR} \\ 2 &: \quad \text{BBRRR}, \quad \text{BRBRR} \\ 3 &: \quad \text{BBBRR}, \quad \text{BBRBR} \\ 4 &: \quad \text{BBBBR} \\ 5 &: \quad \text{BBBBB} \end{aligned}$$

1536. We can factorise directly:

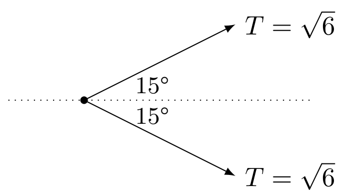
$$\begin{aligned} & (3x-1)^2 - (3x-1)^3 = 0 \\ \Longrightarrow\ & (3x-1)^2\big(1-(3x-1)\big) = 0 \\ \Longrightarrow\ & x = \tfrac{1}{3}, \tfrac{2}{3}. \end{aligned}$$

1537. The boundaries for the first set are at $\pm\sqrt{20}$, where $\sqrt{20}$ lies between 4 and 5. The boundary for the second set is at $\sqrt[3]{50}$, which lies between 3 and 4. This gives

$$\begin{aligned} & \{-4,-3,...,3,4\} \cap \{...,1,2,3\} \\ = & \{-4,-3,-2,-1,0,1,2,3\} \end{aligned}$$

The set contains 8 elements.

1538. The forces exerted by the string are:



Perpendicular to the line of symmetry, there is no resultant component of force. Parallel to it, the forces combine to give

$$\begin{aligned} & 2 \times \sqrt{6}\cos 15° \\ = & \frac{2\sqrt{6}\big(\sqrt{6}+\sqrt{2}\big)}{4} \\ = & \frac{6+2\sqrt{3}}{2} \\ = & 3+\sqrt{3}\text{ N}, \quad \text{as required.} \end{aligned}$$

1539. True. In such a one-tail test, the critical region is of the form $c \le x$. The value c is chosen such that $\mathbb{P}(c \le x) < 0.01$ (getting an outcome in the region $c \le x$ is less probable than 1%), and $\mathbb{P}(c-1 \le x) > 0.01$ (getting an outcome in the larger region $x \ge c-1$ is more probable than 1%).

1540. Since the locus is a circle, we can test distances to the centre. Completing the square for x and y,

$$(x+3)^2 + (y+1)^2 = 30.$$

So, the centre is $(-3,-1)$ and the radius is $\sqrt{30}$. The distances from the centre are $\sqrt{26}$ and $\sqrt{20}$. Since both points lie *inside* the circle, the point further from the centre is closer: this is $(2,-2)$.

1541. Expanding, $3p^2 + 2pq\sqrt{15} + 5q^2 = 155 - 40\sqrt{15}$. Since $p, q \in \mathbb{Z}$, we can equate the rationals and coefficients of $\sqrt{15}$, giving simultaneous equations:

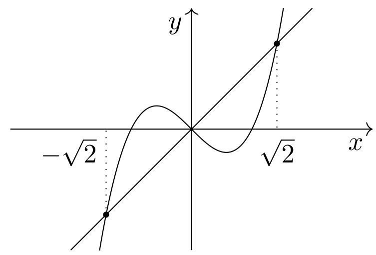
$$\begin{aligned} 3p^2 + 5q^2 &= 155 \\ 2pq &= -40. \end{aligned}$$

Substituting $q = -\frac{20}{p}$ gives

$$\begin{aligned} & 3p^2 + 5 \times \frac{400}{p^2} = 155 \\ \Longrightarrow\ & 3p^4 - 155p^2 + 2000 = 0 \\ \Longrightarrow\ & (3p^2-80)(p^2-25) = 0. \end{aligned}$$

The first bracket gives non-integers, so $p = \pm 5$. Substituting yields $q = \mp 4$.

1542. The derivative is $\frac{dy}{dx} = 3x^2 - 1$, so $m_{\text{tangent}} = -1$ at the origin, giving $m_{\text{normal}} = 1$. The equation of the normal is $y = x$. To find intersections, then, we solve $x^3 - x = x$, giving $x = 0, \pm\sqrt{2}$.

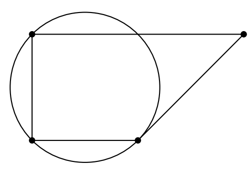


Over the interval $x \in \big(0, \sqrt{2}\big)$, the normal is above the curve, so the y difference between the curves is $x - (x^3 - x)$, which is $2x - x^3$. So, the area is

$$\begin{aligned} A &= \int_0^{\sqrt{2}} 2x - x^3\, dx \\ &= \Big[x^2 - \tfrac{1}{4}x^4\Big]_0^{\sqrt{2}} \\ &= (2-1) - (0) \\ &= 1. \end{aligned}$$

By symmetry, the other region has the same area.

1543. For a counterexample, consider the vertices of a square. Since three of these points define a circle, moving one of the vertices away from the centre will produce a non-cyclic quadrilateral:



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1544. (a) The RHS is $3^x \cdot 3^{2x} = 3^{3x}$, so $y = 3x$.
(b) The RHS is $e^{x-(2x-5)} = e^{-x+5}$, so $y = -x + 5$.

1545. (a) $-a$ is a root of g, since $\mathrm{g}(-a) = 0$,
(b) $-a$ is a fixed point of f, since $\mathrm{f}(-a) = -a$,
(c) a is a root of g^2, since $\mathrm{g}(\mathrm{g}(a)) = \mathrm{g}(-a) = 0$.

1546. (a) Differentiating the area with respect to time, we need the product rule, which gives the sum of the components of the rate of change of area due to changes in x and changes in y:

$$\begin{aligned}\frac{dA}{dt} &= \frac{dx}{dt}y + x\frac{dy}{dt} \\ &= 1 \times y - x \times 1 \\ &= y - x, \text{ as required.}\end{aligned}$$

(b) Substituting the initial conditions $x = 2$ and $y = 10$, the rate of change of area is

$$\left.\frac{dA}{dt}\right|_{t=0} = 10 - 2 = 8.$$

This is positive, so area increases initially.

(c) The area is stationary when $\frac{dA}{dt} = y - x = 0$, i.e. when $x = y$. Since x is increasing from 2 and y is decreasing at the same rate from 10, they are equal at $x = y = 6$, when the area is 36 square units.

1547. Multiplying up by the denominators,

$$\begin{aligned}&\frac{x}{x+1} - \frac{x-1}{x} = \frac{1}{12} \\ \Longrightarrow\ & 12x^2 - 12(x-1)(x+1) = x(x+1) \\ \Longrightarrow\ & 0 = x^2 + x - 12 \\ \Longrightarrow\ & x = -4, 3.\end{aligned}$$

1548. Since each region borders every other region, the probability is zero. You need four colours to have no two like regions bordering each other.

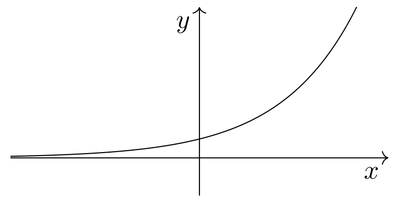
1549. (a) The formula for the velocity V is linear in T, with constant gradient a. Using the formula $y - y_1 = m(x - x_1)$ and the final conditions,

$$\begin{aligned}& V - v = a(T - t) \\ \Longrightarrow\ & V = aT + v - at.\end{aligned}$$

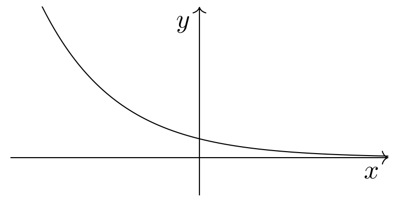
(b) The total displacement, then, is given by the definite integral between $T = 0$ and $T = t$:

$$\begin{aligned}s &= \int_{T=0}^{T=t} aT + v - at \, dT \\ &\equiv \left[\tfrac{1}{2}aT^2 + vT - atT\right]_{T=0}^{T=t} \\ &\equiv \left(\tfrac{1}{2}at^2 + vt - at^2\right) - (0) \\ &\equiv vt - \tfrac{1}{2}at^2, \text{ as required.}\end{aligned}$$

1550. (a) Exponential growth, such as the curve, $y = e^x$ has positive first and second derivatives:



(b) Exponential decay, such as the curve $y = e^{-x}$, has negative first, positive second derivative:

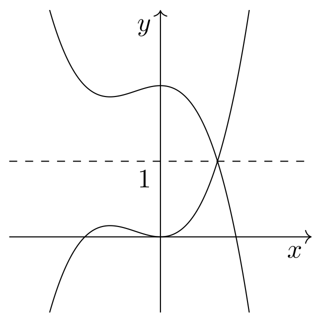


1551. Using laws of logarithms and then the definition of the natural logarithm as $\ln x \equiv \log_e x$,

$$\begin{aligned}&\ln\left(2\sqrt{e^n}\right) - \ln 2 \\ \equiv\ & \ln \frac{2e^{\frac{1}{2}n}}{2} \\ \equiv\ & \ln e^{\frac{1}{2}n} \\ \equiv\ & \tfrac{1}{2}n.\end{aligned}$$

1552. (a) To find stationary points, we set $3x^2 + 2x = 0$, giving $(0, 0)$ and $\left(-\frac{2}{3}, \frac{4}{27}\right)$. Since both of its turning points are below $y = 1$, the cubic can cross $y = 1$ only once.

(b) The cubic is $y = x^2(x + 1)$, with SPs as in part (a). So, the sketch is



(c) Enacting the reflection in the x axis gives $y = -x^3 - x^2$. And the translation vector is then $2\mathbf{j}$. So, the reflected cubic has equation $y = -x^3 - x^2 + 2$.

1553. (a) A circle centred on the origin with radius r has equation $y^2 = r^2 - x^2$. So, the integrand is πy^2. This is the area of a circular cross-section through a sphere of radius r. The integral adds up all such discs, from $-r$ to r, to produce the volume of the sphere.

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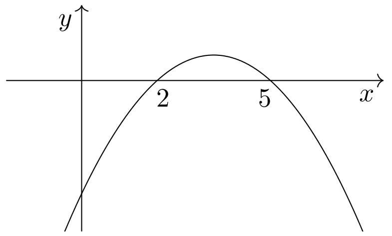
(b) Integrating definitely,

$$\begin{aligned} V &= \int_{-r}^{r} \pi\big(r^2 - x^2\big)\,dx \\ &\equiv \pi\Big[r^2x - \tfrac{1}{3}x^3\Big]_{-r}^{r} \\ &\equiv \pi\Big(\big(r^3 - \tfrac{1}{3}r^3\big) - \big(-r^3 + \tfrac{1}{3}r^3\big)\Big) \\ &\equiv \tfrac{4}{3}\pi r^3, \text{ as required.} \end{aligned}$$

1554. The area under the quadratic has a maximum, so $y = \mathrm{f}(x)$ is a negative parabola. Furthermore, it must have roots at $x = 2$ and $x = 5$, so that any domain larger than $[2, 5]$ causes a reduction in the value of the definite integral. Hence, the sketch is



1555. The equation is factorised, so we can split it. The first factor gives

$$\tan x = \tfrac{\sqrt{3}}{3} \implies x = \tfrac{\pi}{6}, \tfrac{7\pi}{6}.$$

The second factor gives

$$\cos x = \tfrac{\sqrt{2}}{2} \implies x = \tfrac{\pi}{4}, \tfrac{7\pi}{4}.$$

So, the solution is $x \in \big\{\tfrac{\pi}{6}, \tfrac{\pi}{4}, \tfrac{7\pi}{6}, \tfrac{7\pi}{4}\big\}$.

1556. The claim isn't true: if an object is also being held up by, say, tension in a rope, then the magnitude of the reaction will be smaller than the magnitude of the weight.

The justification with NIII is, therefore, incorrect. A weight (gravitational) force could never be the NIII pair of a reaction (contact) force:

- the NIII pair of gravity is always gravity (one on object A, one on object B, the Earth);
- the NIII pair of a contact force is always a contact force (one on object A, one on object B, the ground).

In situations in which the initial claim is correct, i.e. for objects in equilibrium with only weight and reaction acting on them, the justification for these forces being equal in magnitude is, in fact, NI/NII: the resultant force must be zero.

— NOTA BENE —

Again (I can't stress this enough!), the words have *changed meanings* since Newton's day. This is why it is a bad idea (these days, that is, not for Sir Isaac himself) to take NIII in the old form

> *Every action has an equal and opposite reaction.*

That form is actively confusing now. In modern parlance, if one force is a reaction, then its NIII pair is also a reaction. A less snappy, but much more reliable form for the third law is:

> NIII: *Every interaction is modelled with a pair of equal and opposite forces.*

1557. To find the inverse function g^{-1}, we set the output to y and rearrange:

$$\begin{aligned} y &= \frac{2}{\sqrt{3-2x}} + 1 \\ \implies \sqrt{3-2x} &= \frac{2}{y-1} \\ \implies 3 - 2x &= \frac{4}{(y-1)^2} \\ \implies x &= \tfrac{3}{2} - \frac{2}{(y-1)^2}. \end{aligned}$$

Switching inputs and outputs, the inverse is

$$\mathrm{g}^{-1} : x \mapsto \tfrac{3}{2} - \frac{2}{(x-1)^2}.$$

1558. (a) Solving for the x intercepts,

$$\begin{aligned} &\frac{1}{\sqrt{x}} + x - 2 = 0 \\ \implies\ & x^{\frac{3}{2}} - 2x^{\frac{1}{2}} + 1 = 0. \end{aligned}$$

This is a cubic in $x^{\frac{1}{2}}$, with a root at $x^{\frac{1}{2}} = 1$. So, we can factorise:

$$\begin{aligned} &\big(x^{\frac{1}{2}} - 1\big)\big(x + x^{\frac{1}{2}} - 1\big) = 0 \\ \implies\ & x^{\frac{1}{2}} = 1 \text{ or } x^{\frac{1}{2}} = \tfrac{1}{2}\big(-1 + \sqrt{5}\big). \end{aligned}$$

We have ignored the negative root of the quadratic. Squaring, $x = 1$ or $x = \frac{1}{2}(3 - \sqrt{5})$.

(b) Integrating definitely between the x intercepts,

$$\begin{aligned} &\int_{\frac{1}{2}(3-\sqrt{5})}^{1} \frac{1}{\sqrt{x}} + x - 2\,dx \\ &= \Big[2x^{\frac{1}{2}} + \tfrac{1}{2}x^2 - 2x\Big]_{\frac{1}{2}(3-\sqrt{5})}^{1} \\ &= \tfrac{1}{2} - 0.54508... \\ &= -0.04508... \end{aligned}$$

The signed area is negative, because it is below the x axis. The area, therefore, is 0.0451 (3sf).

1559. This is true. The boundary condition of the LHS is $\mathbb{P}(A \mid B) = \mathbb{P}(A)$, which is exactly the condition for independence of events A and B. The same is true of the RHS. Hence, we know that

$$\mathbb{P}(A \mid B) = \mathbb{P}(A) \iff \mathbb{P}(A \mid B') = \mathbb{P}(A).$$

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Consider departure from equality. If knowing that B occurs *increases* the probability of A (LHS of the original inequality), then knowing that B doesn't occur must *decrease* the probability of A (RHS of the original inequality) and vice versa. Hence, the result holds.

1560. Taking natural logarithms of both sides,

$$\begin{aligned} & y = ae^{kx} \\ \implies & \ln y = \ln\left(ae^{kx}\right) \\ \implies & \ln y = \ln a + \ln\left(e^{kx}\right) \\ \implies & \ln y = \ln a + kx. \end{aligned}$$

This is a linear relationship between $\ln y$ and x. The gradient is k and the constant is $\ln a$.

1561. (a) Since p is the only value for which $f(p) = 2$, we have $4x - 1 = p$, therefore $x = \frac{1}{4}(p + 1)$.

(b) Since the equation is already factorised, either $f(x + 1) = 2$ or $f(x) = 3$. Hence, the solution is $x = p - 1, q$.

1562. Choosing points close to the y intercept at $(0, 1)$, counterexamples are:

(a) $(0.2, 0.9)$: above $x + y = 1$, below $x^3 + y^3 = 1$.

(b) $(-0.2, 1.1)$: above $x^3+y^3 = 1$, below $x+y = 1$.

1563. To show $\mathbb{P}(X^2 + 1 > X) = 1$, we need to show that $X^2 + 1 > X$ always holds. The boundary is

$$X^2 - X + 1 = 0.$$

This has $\Delta = -3 < 0$. So, $X^2 + 1$ is never equal to X. Furthermore, since $X^2 + 1$ is a *positive* quadratic, it must always be *greater* than X. So, $\mathbb{P}(X^2 + 1 > X) = 1$, as required.

1564. We need $(x^2 + 1)^2$ to match the x^4 term, which gives $x^4 + 2x^2 + 1$. So, we subtract $(x^2 + 1)$ to match the x^2 term. The constant term is then 1:

$$x^4 + x^2 + 1 \equiv (x^2 + 1)^2 - (x^2 + 1) + 1.$$

1565. (a) Taking u to be the initial velocity and a to be the constant acceleration, we get simultaneous equations, each using $s = ut + \frac{1}{2}at^2$:

$$\begin{aligned} t \in [0, 2] : &\quad 2 = 2u + 2a \\ t \in [0, 4] : &\quad 6 = 4u + 8a \end{aligned}$$

Solving these, $u = \frac{1}{2}$ and $a = \frac{1}{2}$.

(b) Using the values above and the initial position $x = 2$, the formula is $x = 2 + \frac{1}{2}t + \frac{1}{4}t^2$.

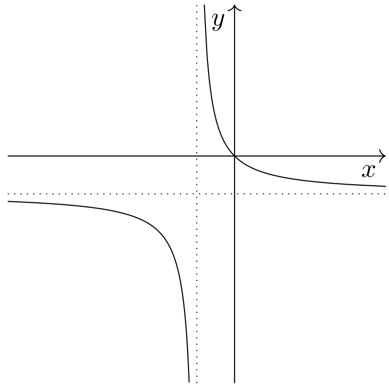
(c) At $t = 10$, $x = 32$. Hence, the displacement is $s = 30$, giving an average speed of 3.

1566. The coefficient -15 is ${}^5\mathrm{C}_4 a$, which gives $a = -3$. Then, the coefficient of x^3 is ${}^5\mathrm{C}_3 a^2$, so $b = 90$.

1567. (a) Using the standard result for the distribution of the sample mean, $\bar{Y} \sim N(100, 2)$.

(b) Calculating a cumulative normal distribution, $\mathbb{P}(99 \le \bar{Y} \le 101) = 0.52049... = 0.520$ (3sf).

1568. The graph $xy = 1$ is the standard reciprocal. We have replaced x by $(x + 1)$ and y by $(y + 1)$, thus enacting translation by vector $\binom{-1}{-1}$. The resulting graph passes through the origin.



1569. Combining the constants of integration into one $+c$ on the left-hand side, and using the standard integral of the natural logarithm,

$$\begin{aligned} & \int 3x(x + 2)\, dx = \int \frac{1}{y}\, dy \\ \implies & x^3 + 3x^2 + c = \ln|y|. \end{aligned}$$

Since $y > 0$, we can dispense with the mod sign:

$$\begin{aligned} & \ln y = x^3 + 3x^2 + c \\ \implies & y = e^{x^3+3x^2+c} \\ \implies & y = e^c \cdot e^{x^3+3x^2}. \end{aligned}$$

Renaming e^c as a (positive) constant A,

$$y = Ae^{x^3+3x^2}, \text{ as required.}$$

1570. If f' is quadratic, then f is a cubic function of the form $f(x) = ax^3 + bx^2 + cx + d$, where $a \neq 0$.

1571. The first two lines have gradient $\frac{1}{3}$, the last two have gradient -3. Hence, the shape is a rectangle. One vertex is at the origin, so we need only find the two adjacent vertices, which are at $(3, 1)$ and $(1, -3)$. Each is situated a distance $\sqrt{10}$ from the origin, proving that the shape enclosed is a square of area 10 square units.

1572. Converting the information into equations,

$$\begin{aligned} 25 &= 8a + b, \\ 93 &= 25a + b. \end{aligned}$$

Solving simultaneously, $a = 4$ and $b = -7$.

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1573. Perpendicular to the acceleration, $8\sin\theta - 4 = 0$, where θ is the acute angle between the 8 N force and the acceleration. This gives $\theta = 30°$. Hence, the obtuse angle between the 8 N and 2 N forces is 150°. Resolving parallel to the acceleration, $8\cos 30° - 2 = 2a$, which gives $a = 2\sqrt{3} - 1$.

1574. Statements (b) and (c) are false.

(b) The counterexample is $a = b$, $c = d$, for which the division on the RHS is undefined, not zero.

(c) The counterexample is $a \neq b$, $c = d$.

1575. The range of sine is $[-1, 1]$. This means that the inputs of these sine functions are irrelevant. The furthest the curve could be from the origin is $\sqrt{1^2 + 1^2} = \sqrt{2}$, which is the radius of the given circle. Hence, the result is proved.

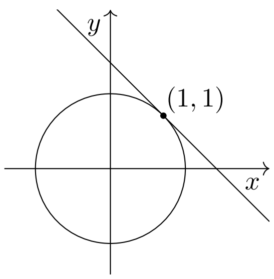
1576. (a) Using the cumulative distribution function on a calculator, assuming $X \sim N(66.1, 14.6^2)$,

i. $\mathbb{P}(X < 30) = 0.00671$ (3sf),

ii. $\mathbb{P}(50 < X < 70) = 0.470$ (3sf).

(b) There are many reasons! Firstly, the sample is too small to give reliable estimates for μ and σ. Secondly, and of broader application, the default position is that a normal distribution should **not** be used, and you need very good reasons for supposing that one does apply. In this case, we have the opposite. For example, since there are upper and lower bounds (whose origins are different) on age, the distribution of the population of ages of walk-in patients is guaranteed not to be symmetrical.

1577. The relevant boundary graphs are a straight line $x + y = 2$ and a circle $x^2 + y^2 = 2$. These are tangent to one another at the point $(1, 1)$.



The line is above and to the right of the circle. Hence, if we are above and to the right of the line, i.e. if the first inequality holds, then we must be outside the circle, i.e. the second inequality must also hold.

1578. To factorise this, we first factorise

$$39867x^2 - 93574x - 57893.$$

Equating this to zero, we solve with the formula. This gives $x = \frac{277}{97}$ or $x = -\frac{209}{411}$. Converting these using the factor theorem, we have

$$(97x - 277)(411x + 209).$$

A quick check that $97 \times 411 = 39867$ shows that no constant factors are needed. Reinstating the y's, the factorisation is

$$(97x - 277y)(411x + 209y).$$

1579. The gradient of the tangent is $2x\,|_{x=p} = 2p$. Then, using $y - y_1 = m(x - x_1)$, the equation of a general tangent is

$$\begin{aligned} & y - p^2 = 2p(x - p) \\ \Longrightarrow\ & y = 2px - 2p^2 + p^2 \\ \Longrightarrow\ & y = 2px - p^2, \text{ as required.} \end{aligned}$$

1580. Taking $t = 0$ to be the time at which the second projectile is dropped, the first projectile has fallen for $t + 1$ seconds at time t. Assuming projectile motion, the difference in vertical position, then, is

$$\tfrac{1}{2}g(t+1)^2 - \tfrac{1}{2}gt^2 \equiv \tfrac{1}{2}g(t+1).$$

This is a linear function of t; it grows, therefore, according to the model, without bound.

1581. Logarithms are undefined for negative inputs. But $1 + x + x^2$ is a positive quadratic with discriminant $-3 < 0$. Hence, its value is always positive. The overall function, therefore, is well defined over $\mathbb{R}$.

1582. (a) Since $\mathbf{p}$ and $\mathbf{q}$ share no components, they are perpendicular. Hence, the angle between them is a right angle.

(b) Using Pythagoras's theorem, this equation is $12 = k\sqrt{1^2 + 2^2 + 2^2}$, giving $k = 4$.

1583. The possibility space has $6 \times 12 = 72$ outcomes, which we can visualise as a rectangular grid. The successful region is a triangle of $1+2+3+4+5 = 15$ outcomes. So, the probability is $p = \frac{15}{72} = \frac{5}{24}$.

1584. Using the chain rule,

$$\begin{aligned} & y = \tan px \\ \Longrightarrow\ & \frac{dy}{dx} = p\sec^2 px. \end{aligned}$$

Then, using the second Pythagorean trig identity $1 + \tan^2 x \equiv \sec^2 x$, we can express this as

$$\frac{dy}{dx} = p + p\tan^2 px.$$

Substituting $y = \tan px$ yields the required result:

$$\frac{dy}{dx} = p + py^2.$$

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1585. The pupil has begun with a statement of what he is trying to prove. That's wrong. Corrected, the statement is "Assume, for a contradiction, that $\sqrt{2}$ **can** be written in the form p/q, where p and q are integers with $\text{hcf}(p, q) = 1$."

— Nota Bene —

A proof by contradiction starts by assuming the opposite of the result to be proved. The aim is then to reach something impossible. For example, in a non-mathematical context, I wish to prove that I am not a parrot. The broad idea is:

If I was a parrot, then I'd be able to fly. I can't fly, so I must not be a parrot.

In the language of proof by contradiction:

Assume, for a contradiction, that I am a parrot. Parrots can fly. So, since I am a parrot, I can fly. But I cannot fly. This is a contradiction. So, I am not a parrot. QED.

1586. Defining x and $\frac{1}{2}P - x$ as the side lengths of the rectangle, the area is given by

$$A = x\left(\tfrac{1}{2}P - x\right) = \tfrac{1}{2}Px - x^2.$$

Setting the derivative to zero,

$$\begin{aligned} \frac{dA}{dx} &= \tfrac{1}{2}P - 2x = 0 \\ \Longrightarrow x &= \tfrac{1}{4}P. \end{aligned}$$

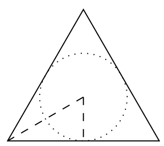
So, the area is optimised with all of the sides of the rectangle equal. This is a square, as required.

1587. (a) Substituting the proposed function,

$$\begin{aligned} & \int_0^x t^k \, dt \equiv \frac{2}{x^k} \\ \Longrightarrow\ & \left[\tfrac{1}{k+1} t^{k+1}\right]_{t=0}^{t=x} \equiv 2x^{-k} \\ \Longrightarrow\ & \tfrac{1}{k+1} x^{k+1} \equiv 2x^{-k}. \end{aligned}$$

(b) Since this is an identity, the two sides must be equivalent for all values of x. Substituting $x = 0$ gives $\frac{1}{k+1} = 2$, hence $k = -\frac{1}{2}$. This can easily be verified to satisfy the identity for all values of x.

1588. The length scale factor between the two triangles is that between the centre-and-vertex and centre-and-midpoint of an equilateral triangle:



The length scale factor is $\sin 30° = \frac{1}{2}$, so the area scale factor is $\frac{1}{4}$. This, therefore, is the area.

1589. In the first three, the range transforms linearly and positively, so we need only transform the lower bound. In (d), the range is also reversed.

(a) $[2, \infty)$

(b) $[5, \infty)$

(c) $[a + b, \infty)$

(d) $(-\infty, a + b]$

1590. In general, two particles can only be considered as a single system in $F = ma$ if they have the same acceleration. In this question, this is not true if the string goes slack. So the first student is right only some of the time. But this cannot happen if the connector is rigid, so the second student is fully correct.

1591. Differentiating with respect to y,

$$\begin{aligned} x &= y^2 + 2y - 6 \\ \Longrightarrow \frac{dx}{dy} &= 2y + 2. \end{aligned}$$

We now reciprocate, giving

$$\frac{dy}{dx} = \frac{1}{2y + 2}.$$

Substituting $y = -2$ gives $m_{\text{tan}} = -\frac{1}{2}$. So, $m_{\text{nor}} = 2$. The normal at $(-6, -2)$ is then

$$\begin{aligned} y + 2 &= 2(x + 6) \\ \Longrightarrow y &= 2x + 10. \end{aligned}$$

— Alternative Method —

Differentiating with respect to y,

$$\begin{aligned} x &= y^2 + 2y - 6 \\ \Longrightarrow \frac{dx}{dy} &= 2y + 2. \end{aligned}$$

Evaluating at $y = -2$, we have $M_{\text{tan}} = -2$, where M is a rate of change of x with respect to y. This gives $M_{\text{nor}} = \frac{1}{2}$. We now use

$$x - x_1 = M(y - y_1).$$

At point $(-6, -2)$, this is

$$\begin{aligned} x + 2 &= \tfrac{1}{2}(y + 6) \\ \Longrightarrow y &= 2x + 10. \end{aligned}$$

1592. (a) Since probabilities are positive and sum to 1, $4p^2 + 3p = 1$, giving $p = \frac{1}{4}$, $q = \frac{1}{5}$.

(b) Restricting the possibility space to the first and third branches,

$$\mathbb{P}(X \mid Y) = \frac{4p^2 \times 2q}{4p^2 \times 2q + 3p \times q} = \tfrac{2}{5}.$$

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1593. Multiplying up, we require

$$1 \equiv A(x^2 - 1) + Bx^2.$$

Equating the coefficients of x^2 gives $A + B = 1$. Equating the constant terms gives $1 = -A$. Hence, $A = -1$ and $B = 2$.

1594. For intersections,

$$\begin{aligned} & x^2 = 8 - (x-4)^2 \\ \Longrightarrow\ & 2x^2 - 8x + 8 = 0 \\ \Longrightarrow\ & (x-2)^2 = 0. \end{aligned}$$

Since $x = 2$ is a double root, the curves cannot cross at $x = 2$; they must just touch. Hence, the two parabolae are tangent at that point.

1595. Squaring and subtracting,

$$R^2(\sec^2\theta - \tan^2\theta) = 13^2 - 5^2 = 144.$$

Using the second Pythagorean trig identity, the trigonometric factor is 1, so $R = \pm 12$.

1596. If f and g are distinct, then E has solution set S. However, if f and g are identical, then E has solution set $\mathbb{R}$.

(a) False.

(b) True.

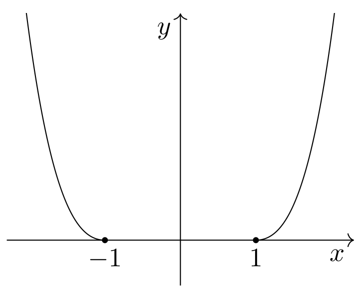
(c) False.

1597. Equating the differences,

$$\begin{aligned} & 2\sqrt{q} + 1 - (\sqrt{q} - 2) = q - (2\sqrt{q} + 1) \\ \Longrightarrow\ & q - 3\sqrt{q} - 4 = 0 \\ \Longrightarrow\ & (\sqrt{q} - 4)(\sqrt{q} + 1) = 0 \\ \Longrightarrow\ & \sqrt{q} = 4, -1. \end{aligned}$$

There is no q with $\sqrt{q} = -1$, so $q = 16$.

1598. Squaring $\sqrt{y} = x^2 - 1$ gives $y = (x^2 - 1)^2$. So, the new curve must be a subset of the given curve. However, x values with $x^2 - 1 < 0$ do not produce points: $\sqrt{y}$ cannot be negative. Hence, the middle section of the curve is missing.



1599. This is correct. The argument can be seen with the car floating in space or standing on ice. The engine still whirrs round, but there is nothing to push the wheels forward. It is the frictional force of the road on the wheels (driving force) that, in the sense of the Newtonian system, does the driving.

1600. A counterexample is the two samples $\{0, 0, 0, 0\}$ and $\{1, 1, 1, 1\}$. Standard deviation is zero for each sample, but non-zero for their combination.

— Nota Bene —

If the means of the two samples are the same, then the result in this question does hold.

— End of 16th Hundred —